

PONTRYAGIN SPACE STRUCTURE IN REPRODUCING KERNEL HILBERT SPACES OVER \ast -SEMIGROUPS

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ABSTRACT. The geometry of spaces with indefinite inner product, known also as Krein spaces, is a basic tool for developing Operator Theory therein. In the present paper we establish a link between this geometry and the algebraic theory of \ast -semigroups. It goes via the positive definite functions and related to them reproducing kernel Hilbert spaces. Our concern is in describing properties of elements of the semigroup which determine shift operators which serve as Pontryagin fundamental symmetries.

INTRODUCTION

There are two ways of looking at \ast -semigroups and positive definite functions defined on them. The first consists in intense analysis of the algebraic structure of a semigroup so as to establish conditions on it, which ensure prescribed properties to hold for any positive definite functions. One of the properties frequently considered is representing each of the positive definite functions as moments of a positive measure. This attitude has been successfully undertaken by Bisgaard resulting in a considerable number of papers, see in particular [4, 5, 6, 7, 8, 9] and references therein, some of them we are going to exploit here. The other way is to determine which of the positive definite functions posses desired properties; a typical example within this category is to detect multidimensional moment sequences.

In the present paper we are going to develop the first thread putting forward the following problem. Impose necessary and sufficient conditions on a distinguished element u of a \ast -semigroup S to generate a Pontryagin fundamental symmetry of any reproducing kernel Hilbert space over the semigroup in question; the precise formulation is exposed as (P), p. 2. Surprisingly, our problem has found a simple algebraic solution. In the context of \ast -separative commutative semigroups the condition on u is: $u = u^\ast$, $u + u = 0$ and $u + s \neq s$ for only a finite number of $s \in S$ (see Proposition 1, Theorem 6).

Let us mention that \ast -semigroups and positive definite functions on them have been originated by Sz.-Nagy in his famous Appendix [19]. He uses the reproducing kernel Hilbert space factorization to prove his "general dilation theorem" for operator valued functions. Since then the RKHS technique has been used from time to time for proving results with dilation flavour behind the screen. We are going to follow suit here.

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1. SHIFT OPERATORS CONNECTED WITH POSITIVE DEFINITE FUNCTIONS –
FORMULATION OF THE PROBLEM

By a $*$ -semigroup we understand a commutative semigroup with an involution, not necessarily having the neutral element. The involution is always denoted by the symbol “ $*$ ” and the semigroup operation is always written in an additive way. In the case when the $*$ -semigroup S has a neutral element 0 we say that $\phi : S \rightarrow \mathbb{C}$ is *positive definite* (we write $\phi \in \mathfrak{P}(S)$) if for every $N \in \mathbb{N} := \{0, 1, \dots\}$ and every $s_0, \dots, s_N \in S$, $\xi_0, \dots, \xi_N \in \mathbb{C}$ we have $\sum_{i,j=0}^N \xi_i \bar{\xi}_j \phi(s_j^* + s_i) \geq 0$. With such ϕ we link a reproducing kernel Hilbert space $\mathcal{H}^\phi \subseteq \mathbb{C}^S$ with a reproducing kernel defined by $K^\phi(s, t) := \phi(t^* + s)$, $t, s \in S$ (see e.g. [2, p.81]). For $s \in S$ we set $K_s^\phi := K(\cdot, s)$ ($s \in S$), it is known that $K_s^\phi \in \mathcal{H}^\phi$ and $f(s) = \langle f, K_s^\phi \rangle$ for every $f \in \mathcal{H}^\phi$. The set $\mathcal{D}^\phi := \text{lin} \{K_s^\phi : s \in S\}$ is dense in \mathcal{H}^ϕ .

For an element $u \in S$ we define the *shift operator* (sometimes called also the translation operator) $A(u, \phi) : \mathcal{D}^\phi \rightarrow \mathcal{D}^\phi$, by

$$A(u, \phi) \left(\sum_j \xi_j K_{s_j}^\phi \right) := \sum_j \xi_j K_{s_j+u}^\phi.$$

It can be shown that $A(u, \phi)$ is well defined, linear and closable (see e.g. [15, Proposition, p.253], [2, p.90]). Our main object will be the closure of the above operator, denoted below by u_ϕ . It is a matter of simple verification that \mathcal{D}^ϕ is contained in the domain of $(u_\phi)^*$ (the adjoint of the operator u_ϕ) and $(u_\phi)^* f = (u^*)_\phi f$ for $f \in \mathcal{D}^\phi$.

Under the above circumstances we can state our problem as follows.

- (P) Provide necessary and sufficient conditions on the element $u \in S$ for the operator u_ϕ to be a fundamental symmetry of a Pontryagin space, i.e. to satisfy

$$u_\phi = u_\phi^*, \quad u_\phi^2 = I_{\mathcal{H}^\phi}, \quad \dim \ker(u_\phi + I_{\mathcal{H}^\phi}) < \infty$$

for every $\phi \in \mathfrak{P}(S)$.

Obviously, the condition $u_\phi = u_\phi^*$ together with $u_\phi^2 = I_{\mathcal{H}^\phi}$ imply that u_ϕ must be bounded on \mathcal{H} .

We continue with some basic definitions and notations concerning $*$ -semigroups. Let S and T be $*$ -semigroups, a mapping $\chi : S \rightarrow T$ is called *$*$ -homomorphism* if $\chi(s+t) = \chi(s) + \chi(t)$ for all $s, t \in S$, $\chi(s^*) = (\chi(s))^*$ for all $s \in S$. A *character* on S is a nonzero $*$ -homomorphism $\chi : S \rightarrow \mathbb{C}$ where the latter set is understood as a semigroup with multiplication as the operation and the conjugation as involution. It is obvious that if S has a neutral element 0 then $\chi(0) = 1$ for all $\chi \in S^*$. Let $\mathcal{A}(S^*)$ be the least σ -algebra of subsets of S^* rendering measurable all the functions

$$(1) \quad \hat{s} : S^* \ni \chi \mapsto \chi(s) \in \mathbb{C}, \quad s \in S.$$

If μ is a positive measure on S^* such that all the functions listed in (1) are square-integrable we define a function $\mathcal{L}(\mu) : S \rightarrow \mathbb{C}$ by $\mathcal{L}(\mu)(s) := \int_{S^*} \sigma(s) d\mu(\sigma)$ ($s \in S$). We call $\phi : S \rightarrow \mathbb{C}$ a *moment function* ($\phi \in \mathfrak{M}(S)$) if $\phi = \mathcal{L}(\mu)$ for some measure μ on S^* . It is easy to verify that $\mathfrak{M}(S) \subseteq \mathfrak{P}(S)$, if the latter inclusion is an equality then we call S *semiperfect*. For examples of semiperfect and non-semiperfect $*$ -semigroups and more general concepts of semiperfectness see [4].

We call S **-separative* if the characters separate points in S . The *greatest *-homomorphic *-separative image of S* is the semigroup S/\sim , where the equivalence relation \sim on S is defined by the condition that $s \sim t$ if and only if $\sigma(s) = \sigma(t)$ for all $\sigma \in S^*$; addition and involution in S/\sim are those that make the quotient mapping a *-homomorphism. The elements of S/\sim will be denoted as equivalence classes $[s]$ ($s \in S$). If S has a zero, then we will use the symbol 0 for the neutral element of both S and S/\sim , instead of using $[0]$. Since every character on S generates a character on S/\sim , the latter semigroup is in fact *-separative. The following simple proposition gives answer to the question when the operator u_ϕ defined above is a fundamental symmetry of a Krein space, i.e. when $(u_\phi)^2 = u_\phi$ and $u_\phi = (u_\phi)^*$.

Propositon 1. *Let S be a commutative *-semigroup with zero. For each element $u \in S$ the following conditions are equivalent:*

- (i) $[2u] = 0$ and $[u] = [u^*]$;
- (ii) for every $\phi \in \mathfrak{M}(S)$ we have $(u_\phi)^2 = I_{\mathcal{H}^\phi}$ and $(u_\phi)^* = u_\phi$;
- (iii) for every $\phi \in \mathfrak{P}(S)$ we have $(u_\phi)^2 = I_{\mathcal{H}^\phi}$ and $(u_\phi)^* = u_\phi$;

Moreover, (i) implies that u_ϕ is a bounded, selfadjoint operator on \mathcal{H}^ϕ for every $\phi \in \mathfrak{P}(S)$.

Proof. (i) \Rightarrow (iii) Suppose that (i) holds and let $\phi \in \mathfrak{P}(S)$. For every $\sigma \in S^*$ we have $\sigma(u) = \sigma(u^*)$ and $\sigma(2u) = \sigma(0) = 1$. Consequently, for every $\sigma \in S^*$ and every $s \in S$

$$(2) \quad \sigma(s+u) = \sigma(s+u^*), \quad \sigma(t+2u) = \sigma(t), \quad \sigma(s^*+u^*+u+s) = \sigma(s^*+s).$$

By [6, Thm.2] we have

$$(3) \quad \phi(s+u) = \phi(s+u^*), \quad s \in S,$$

$$(4) \quad \phi(s+2u) = \phi(s), \quad s \in S,$$

$$(5) \quad \phi(s^*+u^*+u+s) = \phi(s^*+s), \quad s \in S,$$

The last of these three equalities, together with [15, Cor.1], implies that the operator u_ϕ is in $\mathbf{B}(\mathcal{H}^\phi)$. It is also selfadjoint, since for $f = \sum_i \xi_i K_{s_i}^\phi \in \mathcal{D}^\phi$ we have

$$\begin{aligned} \langle u_\phi f, f \rangle &= \sum_{i,j} \xi_i \bar{\xi}_j \langle K_{s_i+u}^\phi, K_{s_j}^\phi \rangle = \sum_{i,j} \xi_i \bar{\xi}_j \phi(s_i+u+s_j^*) \stackrel{(3)}{=} \\ &\quad \sum_{i,j} \xi_i \bar{\xi}_j \phi(s_i+u^*+s_j^*) = \langle f, u_\phi f \rangle. \end{aligned}$$

The fact that $(u_\phi)^2 = I_{\mathcal{H}^\phi}^\phi$ can be obtained similarly as selfadjointness of u_ϕ , with the use of (4) instead of (3). This finishes the proof of (iii) and of the ‘Moreover’ part of the proposition.

The implication (iii) \Rightarrow (ii) is trivial. The proof (ii) \Rightarrow (i) goes by contraposition. Suppose first that $[2u] \neq 0$, i.e. there exists a character σ such that $\sigma(u)^2 = \sigma(2u) \neq \sigma(0) = 1$. We put $\phi := \mathcal{L}(\delta_\sigma)$, where δ_σ stands for the Dirack measure on S^* concentrated in σ . Let $\langle \cdot, - \rangle$ denote the scalar product on \mathcal{H}^ϕ . Observe that

$$\langle K_0^\phi, K_0^\phi \rangle = \phi(0) = \sigma(0) \neq \sigma(2u) = \phi(2u) = \langle K_{2u}^\phi, K_0^\phi \rangle = \langle (u_\phi)^2 K_0^\phi, K_0^\phi \rangle.$$

In consequence, $I_{\mathcal{H}^\phi} \neq (u_\phi)^2$. Similarly, if $\tau(u) \neq \tau(u^*)$ for some $\tau \in S^*$ then for $\psi := \mathcal{L}(\delta_\tau)$ the operator u_ψ is not symmetric in \mathcal{H}^ψ . □

Let us consider now a situation when S and T are $*$ -semigroups with zeros and h is a $*$ -homomorphism from S into T satisfying $h(0) = 0$. Note that if an element $u \in S$ is such that $[2u] = 0$, $[u] = [u^*]$ then $[2h(u)] = 0$, $[h(u)] = [h(u)^*]$. This comes from the fact that for every character σ on T the function $\sigma \circ h$ is a character on S . Observe also that $\phi \circ h \in \mathfrak{P}(S)$ for every $\phi \in \mathfrak{P}(T)$.

Propositon 2. *Assume that S , T and h are as above and that h is additionally onto. Let $u \in S$ be such that $[2u] = 0$, $[u] = [u^*]$ and let $\phi \in \mathfrak{P}(T)$. Then the operators $u_{\phi \circ h}$ in $\mathcal{H}^{\phi \circ h}$ and $h(u)_\phi$ in \mathcal{H}^ϕ are unitarily equivalent.*

Proof. Let $\langle \cdot, - \rangle_\phi$ and $\langle \cdot, - \rangle_{\phi \circ h}$ denote the scalar products on \mathcal{H}^ϕ and $\mathcal{H}^{\phi \circ h}$ respectively. Since

$$\begin{aligned} \left\langle \sum_{j=1}^N \xi_j K_{s_j}^{\phi \circ h}, \sum_{j=1}^N \xi_j K_{s_j}^{\phi \circ h} \right\rangle_{\phi \circ h} &= \sum_{i,j=1}^N \xi_i \bar{\xi}_j (\phi \circ h)(s_i + s_j^*) \\ &= \sum_{i,j=1}^N \xi_i \bar{\xi}_j \phi(h(s_i) + h(s_j^*)) = \left\langle \sum_{j=1}^N \xi_j K_{h(s_j)}^\phi, \sum_{j=1}^N \xi_j K_{h(s_j)}^\phi \right\rangle_\phi, \end{aligned}$$

the condition $V(K_s^{\phi \circ h}) := K_{h(s)}^\phi$ ($s \in S$) properly defines an isometry between $\mathcal{H}^{\phi \circ h}$ and \mathcal{H}^ϕ . Since h is onto, the range of V is dense in \mathcal{H}^ϕ , and so V is a unitary operator. To finish the proof we need to show that

$$(6) \quad h(u)_\phi V f = V u_{\phi \circ h} f, \quad f \in \mathcal{H}^{\phi \circ h}.$$

This can be easily verified for $f \in \mathcal{D}^{\phi \circ h}$. Since all the operators appearing in (6) are bounded, the proof is finished. \square

Applying the above to the quotient semigroup $T = S/\sim$ and h as the quotient mapping, together with the fact from [9] that every $\phi \in \mathfrak{P}(S)$ ($\phi \in \mathfrak{M}(S)$) is of the form $\phi = \psi \circ h$ for some $\psi \in \mathfrak{P}(S/\sim)$ ($\psi \in \mathfrak{M}(S/\sim)$) gives the following.

Corollary 3. *Assume that S is a $*$ -semigroup with zero and $u \in S$ is such that $[2u] = 0$, $[u] = [u^*]$. Then*

$$\begin{aligned} \dim \ker(u_\phi + I) &< +\infty \text{ for every } \phi \in \mathfrak{P}(S) \text{ } (\phi \in \mathfrak{M}(S)) \\ \iff \dim \ker([u]_\psi + I) &< +\infty \text{ for every } \psi \in \mathfrak{P}(S/\sim) \text{ } (\psi \in \mathfrak{M}(S/\sim)) \end{aligned}$$

2. EXAMPLES

Example 4. *Consider the semigroup $S = \mathbb{Z}_2 \times \mathbb{N}$ with standard addition and the identical involution. As usually (cf. [2]) we identify S^* with $\{-1, 1\} \times \mathbb{R}$, note that S is $*$ -separative. The only nonzero element satisfying $u = u^*$, $2u = 0$ is $u = (1, 0)$. Let ϕ be a positive definite mapping, we will now compute the eigenspaces of u_ϕ . Since S is semiperfect ([5]), there exists a Borel measure μ on S^* such that $\phi = \mathcal{L}(\mu)$. Having our interpretation of characters in mind we get*

$$\phi(x, n) = \int_{\mathbb{R}} (-1)^x t^n d\mu_-(t) + \int_{\mathbb{R}} t^n d\mu_+(t), \quad x \in \mathbb{Z}_2, n \in \mathbb{N},$$

with $\mu_\pm := \mu|_{\{\pm 1\} \times \mathbb{R}}$. Let us define the functions $f_{x,n} : S^* \rightarrow S^*$ ($x \in \mathbb{Z}_2, n \in \mathbb{N}$) by

$$f_{x,n}(\varepsilon, t) := \varepsilon^x t^x, \quad \varepsilon \in \{-1, 1\}, t \in \mathbb{R}, x \in \mathbb{Z}_2, n \in \mathbb{N},$$

and note that they all are square integrable. By \mathcal{P}^μ we define the closure in $L^2(\mu)$ of the linear span of the functions $f_{x,n}$ ($x \in \mathbb{Z}_2$, $n \in \mathbb{N}$). The formula

$$(7) \quad V(K_{x,n}^\phi) := f_{x,n}, \quad x \in \mathbb{Z}_2, n \in \mathbb{N},$$

constitutes a unitary isomorphism between \mathcal{H}^ϕ and \mathcal{P}^μ . The shift operator u_ϕ is unitary equivalent (via V) to the following operator M

$$(Mf)(\varepsilon, t) := \varepsilon f(\varepsilon, t), \quad (\varepsilon, t) \in \{-1, 1\} \times \mathbb{R}, f \in \mathcal{P}^\mu.$$

It is not hard to see that

$$\ker(M \pm I) = \{f \in \mathcal{P}^\mu : f(\pm 1, \cdot) = 0 \text{ } \mu_{\pm-a.e}\}.$$

Note that $\dim \ker(u_\phi \pm I) = \dim \ker(M \pm I)$ and the latter is finite dimensional if and only if the support of μ_\mp is a finite set. In particular there exists a mapping $\phi \in \mathfrak{P}(S) = \mathfrak{M}(S)$ such that $\dim \ker(u_\phi + I) = \infty$.

At this point we present a useful for our purposes construction. Let S and T be two disjoint $*$ -semigroups (which only a formal restriction) and let $h : S \mapsto T$ be a $*$ -homomorphism. We endow the set $S \cup T$ with the $*$ -semigroup structure in the following way. The addition on $S \cup T$, denoted by the same symbol '+', is defined by

$$s + t := \begin{cases} s + t & : s, t \in S \text{ or } s, t \in T \\ s + h(t) & : s \in S, t \in T \\ t + h(s) & : t \in S, s \in T \end{cases}$$

The involution on $S \cup T$ (still denoted by ' $*$ ') is such that its restriction to both S and T is the original involution on S and T , respectively. We denote the above constructed semigroup by $U(S, T, h)$. The reader can easily check a general fact, that if S and T are $*$ -separative then $U(S, T, h)$ is $*$ -separative as well.

Example 5. Consider a semigroup $S = U(\mathbb{Z}_2, \mathbb{N}, h_0)$ where $h_0(x) = 0$ ($x \in \mathbb{Z}_2$). The element $0_{\mathbb{Z}_2}$ is the neutral element of S . Take $u := 1_{\mathbb{Z}_2}$, clearly $u = u^*$ and $2u = 0_{\mathbb{Z}_2}$. Let ϕ be any positive definite function on S and suppose that $f \in \ker(u_\phi + I)$. This means that for $n \in \mathbb{N}$

$$f(n) = f(n + u) = \langle f, K_{n+u}^\phi \rangle = \langle f, u_\phi K_n^\phi \rangle = \langle u_\phi f, K_n^\phi \rangle = -f(n),$$

hence $f|_{\mathbb{N}} = 0$. A similar calculation shows that $f(u) = -f(0_{\mathbb{Z}_2})$. Hence, the eigenspace $\ker(e_\phi + I)$ is spanned by the single function

$$f(s) = \begin{cases} 0 & : s \in \mathbb{N} \\ 1 & : s = 0_{\mathbb{Z}_2} \\ -1 & : s = 1_{\mathbb{Z}_2} \end{cases}$$

if $f \in \mathcal{H}^\phi$ or is trivial otherwise. Resuming, $\dim \ker(u_\phi + I) \leq 1$ for all positive definite ϕ .

3. MAIN RESULT

Theorem 6. Let S be a commutative $*$ -semigroup with zero and let $u \in S$ be such that $[2u] = 0$ and $[u] = [u^*]$. Then the following conditions are equivalent:

- (i) the set $\{[s] \in S/\sim : [u + s] \neq [s]\}$ is finite¹;
- (ii) $\sigma(u) = -1$ for only finitely many $\sigma \in S^*$;

¹ Cf. Remark 7.

- (iii) $\dim \ker(u_\phi + I_{\mathcal{H}^\phi}) < \infty$ for all $\phi \in \mathfrak{M}(S)$;
- (iv) $\dim \ker(u_\phi + I_{\mathcal{H}^\phi}) < \infty$ for all $\phi \in \mathfrak{P}(S)$.

Moreover, if (i) holds then $\dim \ker(u_\phi + I_{\mathcal{H}^\phi})$ is less or equal to the half of the number elements of the set mentioned in (i).

Let us stress here the solution of the main problem of the paper, which follows from Proposition 1 and Theorem 6 : *the operator u_ϕ is a fundamental symmetry of a Pontryagin space for every $\phi \in \mathfrak{P}(S)$ (equivalently: $\phi \in \mathfrak{M}(S)$) if and only if $[2u] = 0$, $[u] = [u^*]$ and the set $\{[s] \in S/\sim : [u + s] \neq [s]\}$ is finite.*

Remark 7. Condition (ii) is equivalent to

- (ii') $\sigma([u]) = -1$ for only finitely many characters σ on S/\sim ,

since the characters on S and S/\sim are in one-to-one correspondence. This, together with Corollary 3 (and the remarks above it) allows us to reduce the proof of Theorem 6 to the case when S is $*$ -separative. However, note that the conditions (i)–(iv) are not equivalent to the following:

the set $\{s \in S : u + s \neq s\}$ is finite.

Indeed, consider the following example. Let $S = \mathbb{Z}_4 \times \mathbb{N}$ with the natural operation $+$ and the identical involution. The greatest $*$ -separative homomorphic image of S is ($*$ -isomorphic with) $\mathbb{Z}_2 \times \mathbb{N}$ and the quotient homomorphism maps $u := (2, 0)$ to $(0, 0)$. We have that $u + s \neq s$ for all $s \in S$ but the set mentioned in (i) is empty. It remains an open problem if the condition

the set $\{s \in S : [u + s] \neq [s]\}$ is finite.

is equivalent to (i).

Before the proof we introduce the notion of a $*$ -archimedean component of a semigroup. We call a $*$ -semigroup H (not necessarily with 0) *$*$ -archimedean* if for all $s, t \in H$ there exists $m \in \mathbb{N} \setminus \{0\}$ such that $m(s + s^*) \in t + H$. An *$*$ -archimedean component* of a $*$ -semigroup S is a maximal $*$ -archimedean $*$ -subsemigroup. Though $*$ -archimedean component is $*$ -semigroup for itself it is possible for it not to have the neutral element even if S does. It can be shown that two elements s, t belong to the same $*$ -archimedean component of S if and only if $m(s + s^*) \in t + S$ and $n(t + t^*) \in s + S$ for some $m, n \in \mathbb{N} \setminus \{0\}$. Furthermore, S is the disjoint union of its $*$ -archimedean components, see [12, Section 4.3] for the case of identical involution. The following Lemma was proven in [7] (Lemma 2), the proof for an arbitrary involution requires minimal effort.

Lemma 8. *If H is a $*$ -archimedean component of a $*$ -semigroup S then every character on H is everywhere nonzero and extends to a character on S .*

If H and K are two $*$ -archimedean components of S then $H + K$ is contained in one single $*$ -archimedean component of S . If $(S_i)_{i \in I}$ is the family of all $*$ -archimedean components of S then we define the operation $+$ on I by:

$$i + j = k \text{ if and only if } S_i + S_j \subseteq S_k, \quad i, j, k \in I.$$

Since $S_i + S_i \subseteq S_i$ for all $i \in I$ we have that $i + i = i$. Therefore I is a semilattice with the natural partial order given by the condition that $i \leq j$ if and only if $i + j = j$. The following easy lemma is left as an exercise for the reader.

Lemma 9. *Let S be a $*$ -semigroup with zero and let $(S_i)_{i \in I}$ be the family of all $*$ -archimedean components S . If $u \in S$ be such that $2u = 0$, then u belongs to the same $*$ -archimedean component as 0 . In particular, $u + S_i \subseteq S_i$ for all $i \in I$.*

Proof of Theorem 6. As it was said in Remark 7 we may assume that S is $*$ -separative. To prove (i) \Rightarrow (iv) let us put

$$U := \{s \in S : u + s = s\}$$

and suppose that the set $S \setminus U$ contains only a finite number M of elements. Take $\phi \in \mathfrak{P}(S)$. We show that

$$\dim \ker(u_\phi + I) \leq M/2,$$

this will also prove the last statement of Theorem 6. Let us fix an arbitrary $f \in \ker(u_\phi + I)$. We have

$$f(s + u) = \langle f, u_\phi K_s^\phi \rangle = \langle u_\phi f, K_s^\phi \rangle = -f(s), \quad s \in S.$$

This means that $f|_U \equiv 0$. Observe that the relation

$$aRb \iff (a = u + b \text{ or } a = b)$$

is an equivalence relation on $S \setminus U$ and that the each equivalence class contains exactly two elements. Take any representees $s_1, \dots, s_{M/2}$ of the equivalence classes of R . It is clear, that

$$\ker(u_\phi + I) \subseteq \text{lin} \{\delta_{s_i} - \delta_{s_i + u} : i = 1, \dots, M/2\}.$$

Consequently $\dim \ker(u_\phi + I) \leq M/2$.

(iv) \Rightarrow (iii) is obvious. (iii) \Rightarrow (ii) Suppose that $\{\sigma_n : n \in \mathbb{N}\}$ is an infinite set of characters satisfying $\sigma_n(u) = -1$ ($n \in \mathbb{N}$). We define a measure μ on S^* by $\mu := \sum_{n=0}^{\infty} 2^{-n} \delta_{\sigma_n}$ and we take a mapping $\phi = \mathcal{L}(\mu)$. Note that for every $N \in \mathbb{N}$ and for every $s_0, \dots, s_N \in S$, $\xi_0, \dots, \xi_N \in \mathbb{C}$ we have

$$\begin{aligned} \left| \sum_{j=0}^N \xi_j \sigma_n(s_j) \right|^2 &\leq 2^n \int_{S^*} \left| \sum_{j=0}^N \xi_j \sigma(s_j) \right|^2 d\mu(\sigma) = 2^n \sum_{i,j=0}^N \xi_i \bar{\xi}_j \int_{S^*} \sigma(s_i + s_j^*) d\mu(\sigma) = \\ &= 2^n \sum_{i,j=0}^N \xi_i \bar{\xi}_j \phi(s_i + s_j^*), \quad n \in \mathbb{N}. \end{aligned}$$

By the RKHS Test ([16] and also [18]) we get that $\sigma_n \in \mathcal{H}^\phi$ for $n = 1, 2, \dots$. Now observe that

$$u_\phi(\sigma_n)(s) = \sigma_n(u + s) = -\sigma_n(s), \quad s \in S, n = 1, 2, \dots$$

Therefore $\sigma_n \in \ker(u_\phi + I)$, $n = 1, 2, \dots$. It remains to show that the functions σ_n , $n \in \mathbb{N}$, are linearly independent. But this results from the well known fact that all characters are linearly independent².

² We can use the following argument: For every $s \in S$ the function \hat{s} is a character on the dual semigroup S^* and it is trivial that the family $(\hat{s})_{s \in S}$ separates elements of S^* . Proposition 2 of [11] (see also [2, Proposition 6.1.8]) says that if T is a semigroup and $C \subseteq T^*$ separates points, then the functions $\hat{t}|_C$, $t \in T$ are linearly independent in \mathbb{C}^C . We use this result for $T = S^*$ and $C = \{\hat{s} : s \in S\}$, the functions $\hat{\sigma}|_C$ can be identified with characters on S .

(ii) \Rightarrow (i) Suppose that the number of elements $s \in S$ satisfying $u + s \neq s$ is infinite. We show that there exists infinitely many characters σ on S such that $\sigma(u) = -1$.

Let $(S_i)_{i \in I}$ be the family of all $*$ -archimedean components of S . Set

$$I_0 := \{i \in I : u + s \neq s \text{ for some } s \in S_i\}.$$

Our assumption implies that³ either

$$(8) \quad S_j \text{ is infinite for some } j \in I_0$$

or

$$(9) \quad I_0 \text{ is infinite.}$$

Let us first assume (8). Take $s_0 \in S_j$ such that $u + s_0 \neq s_0$. By Lemma 9 we have $u + s_0 \in S_j$. The $*$ -semigroup S_j is $*$ -separative as a subsemigroup of a $*$ -separative semigroup S . Therefore, there exists a character σ_0 on S_j such that $\sigma_0(u + s_0) \neq \sigma_0(s_0)$. By Lemma 8 σ_0 extends to some character $\tilde{\sigma}_0$ on S . Since $u = u^*$ and $2u = 0$ we have $\tilde{\sigma}_0(u) \in \{-1, 1\}$. But

$$\tilde{\sigma}_0(u)\sigma_0(s_0) = \sigma_0(u + s_0) \neq \sigma_0(s_0).$$

Hence, $\tilde{\sigma}_0(u) = -1$, which means that $\sigma_0(u + s_0) = -\sigma_0(s_0)$.

Denote by A the set of all those characters σ on S_j satisfying $\sigma(u + s_0) = -\sigma(s_0)$. Since σ_0 is everywhere nonzero on S_j (Lemma 8), the mapping

$$S_j^* \ni \sigma \longmapsto \sigma_0 \sigma \in S_j^*$$

is bijective. Moreover, it maps A onto $S_j^* \setminus A$. Since S_j is infinite and $*$ -separative, S_j^* is infinite as well. Hence, A is infinite. By Lemma 8, there is an infinite number of characters σ on S satisfying $\sigma(u + s_0) = -\sigma(s_0)$ and consequently $\sigma(u) = -1$.

Let us assume now (9). For each $i \in I_0$ we take a character σ_i on S satisfying $\sigma_i(u) = -1$, $\sigma_i(s) \neq 0$ for $s \in S_i$ (such a character exist by repeating the proof from the previous case). We also define a family of characters $\chi_i \in S^*$ ($i \in I_0$) by

$$\chi_i(s) = \begin{cases} 1 & \text{if } s \in S_j \text{ and } j \leq i \\ 0 & \text{otherwise} \end{cases}, \quad s \in S, i \in I_0.$$

Finally, we put $\rho_i := \sigma_i \chi_i$ ($i \in I_0$). By Lemma 9 we have that u is in the same $*$ -archimedean component as 0, denote this component by S_{j_0} . It is clear that $j_0 \leq i$ for all $i \in I$, therefore $\chi_i(u) = 1$ and $\rho_i(u) = -1$ for all $i \in I_0$. The only thing that lasts is to show that $\rho_i \neq \rho_j$ for $i \neq j$. If $i \neq j$ then, by symmetry, we can assume that $j \not\leq i$. Thus $\chi_i|_{S_j} = 0$ and $\rho_i|_{S_j} = 0$. But $\chi_j|_{S_j} \equiv 1$ and σ_j is, by definition, everywhere nonzero on S_j . Therefore $\rho_i|_{S_j} \equiv 0 \neq \rho_j|_{S_j}$.

□

Remark 10. The alternative of (8) and (9) in the proof of (ii) \Rightarrow (i) becomes more clear if we observe that $u + s \neq s$ for all $s \in S_i$, provided that $i \leq j \in I_0$. Indeed, suppose that $u + s = s$ for some $s \in S_i$, $u + s_0 \neq s_0$ for some $s_0 \in S_j$ and $i \leq j$. This gives us

$$(u + s_0) + (s + s_0) = u + s + s_0 + s_0 = s_0 + (s + s_0).$$

³ Remark 10 shows that it is even equivalent to

By Lemma 9 we have $u + s_0 \in S_j$. Moreover, $s + s_0 \in S_j$ because $i \leq j$. The semigroup S_j is cancellative as a $*$ -archimedean component of a $*$ -separable group (see [8, p.63]). This gives us $u + s_0 = s_0$, contradiction. The example below shows that both (8) and (9) are possible.

Example 11. Let $S = \mathbb{Z}_2 \times \mathbb{N}$, with the natural addition on \mathbb{Z}_2 and \mathbb{N} and the identical involution. The element $u = (1, 0)$ is like in (8).

Let us now consider the semigroup $T = \mathbb{Z}_2 \times \mathbb{N}$, with the natural addition on \mathbb{Z}_2 and maximum as the operation on \mathbb{N} , the involution is again set to identity. It is easy to see that T is $*$ -separable. The element $u = (1, 0)$ is such that (9) is satisfied. This example shows one more thing. Namely, the condition

$$\dim \ker(u_\phi + I_{\mathcal{H}^\phi}) < \infty \text{ for all } \phi \in \mathfrak{M}(S) \text{ of compact support}$$

is not equivalent to any of the conditions of Theorem 6. Indeed, the characters on T form a discrete, enumerable set. If the mapping $\phi \in \mathfrak{M}(S)$ is compactly supported then it is finitely supported and consequently the space \mathcal{H}^ϕ is finite dimensional. Hence, $u = (1, 0)$ satisfies the above condition, but does not satisfy (i).

Note that in Proposition 1 restricting to compactly supported moment functions is possible because the function ϕ constructed in the proof (ii) \Rightarrow (i) is supported by only one character.

4. FUNCTIONS WITH A FINITE NUMBER OF NEGATIVE SQUARES

The condition $\dim \ker(u_\phi + I_{\mathcal{H}^\phi}) < \infty$ can be written also in the language of negative squares. Precisely speaking, by the number of negative squares of a mapping $\psi : S \rightarrow \mathbb{C}$ satisfying

$$(10) \quad \psi(s) = \overline{\psi(s^*)}, \quad s \in S,$$

we understand the maximum, taken over all numbers $N \in \mathbb{N}$ and all sequences s_0, \dots, s_N , of the number of negative eigenvalues of the symmetric matrix $(\psi(s_i + s_j^*))_{i,j=0}^N$. Note that if $[u] = [u^*]$ and $\phi \in \mathfrak{P}(S)$ then the mapping $\psi := \phi(\cdot + u)$ satisfies (10). Indeed, take any character σ . Then $\sigma(s + u) = \sigma(s)\sigma(u) = \sigma(s)\sigma(u^*) = \sigma(s + u^*)$. By the result of [6] we get $\phi(s + u) = \phi(s + u^*)$. Combining this with $\phi(t) = \overline{\phi(t^*)}$ ($t \in S$) [2, 4.1.6] proofs the claim.

Propositon 12. Let S be a $*$ -semigroup with zero and let $u \in S$ be such that $[2u] = 0$ and $[u] = [u^*]$ and let $\phi \in \mathfrak{P}(S)$. Then the number of negative squares of the mapping $\phi(\cdot + u)$ equals $\dim \ker(u_\phi + I)$. Consequently, conditions (i)–(iv) of Theorem 6 are equivalent to each of the following:

- (v) the mapping $\phi(\cdot + u)$ has a finite number of negative squares for every $\phi \in \mathfrak{P}(S)$;
- (vi) the mapping $\phi(\cdot + u)$ has a finite number of negative squares for every $\phi \in \mathfrak{M}(S)$.

Proof. (cf. [1, 13, 14] for similar arguments) First let us assume that $\dim \ker(u_\phi + I) = m \in \mathbb{N}$. Consider the indefinite inner product space $(\mathcal{H}^\phi, \langle u_\phi \cdot, \cdot \rangle)$. Since \mathcal{D}^ϕ is dense in \mathcal{H}^ϕ we can find elements $s_1, \dots, s_k \in S$ and vectors $\alpha^i = (\alpha_1^i, \dots, \alpha_k^i) \in \mathbb{C}^k$ ($i = 1, \dots, m$) such that the elements

$$(11) \quad f^i := \sum_{j=1}^k \alpha_j^i K_{s_j}^\phi \quad (i = 1, \dots, m)$$

span an m -dimensional negative subspace ([3, Theorem IX.1.4]). Let

$$(12) \quad A := (\phi(s_j + s_{j'}^* + u))_{j,j'=1}^k \in \mathbb{C}^{k \times k}.$$

Note that for $i, l = 1, \dots, m$ we have

$$(13) \quad \langle A\alpha^l, \alpha^i \rangle = \sum_{j,j'=1}^k \alpha_j^l \overline{\alpha_{j'}^i} K^\phi(s_j + u, s_{j'}) = \langle u_\phi f^l, f^i \rangle.$$

Hence, the subspace $\text{lin}\{\alpha^1, \dots, \alpha^k\}$ is a negative subspace of the indefinite inner product space $(\mathbb{C}^m, \langle A \cdot, \cdot \rangle)$. Since f^1, \dots, f^m are linearly independent, the vectors $\alpha^1, \dots, \alpha^m$ are linearly independent as well. Therefore, the matrix A has at least m negative eigenvalues.

Now let us assume that for some choice of $s_1, \dots, s_k \in S$ the matrix A defined as in (12) has m negative eigenvalues. Then there exists linearly independent vectors $\alpha^i = (\alpha_1^i, \dots, \alpha_k^i) \in \mathbb{C}^k$ ($i = 1, \dots, m$) such that

$$(14) \quad \langle A\alpha^l, \alpha^i \rangle = \delta_{il} \lambda_i \quad i, l = 1, \dots, m,$$

with some $\lambda_1, \dots, \lambda_m < 0$. We define f_1, \dots, f_m as in (11) (with the new meaning of s_1, \dots, s_m). Using the calculation in (13) we get that the space $\text{lin}\{f_1, \dots, f_m\}$ is a negative subspace of $(\mathcal{H}^\phi, \langle u_\phi \cdot, \cdot \rangle)$. We show now that f_1, \dots, f_m are linearly independent. If $\sum_{i=1}^m \beta_i f_i = 0$ for some $\beta_1, \dots, \beta_m \in \mathbb{C}$ then, by (13),

$$\left\langle A \sum_{i=1}^m \beta_i \alpha^i, \sum_{i=1}^m \beta_i \alpha^i \right\rangle = \left\langle u_\phi \sum_{i=1}^m \beta_i f_i, \sum_{i=1}^m \beta_i f_i \right\rangle = 0.$$

But A is strictly negative on $\text{lin}\{\alpha^1, \dots, \alpha^m\}$ and $\alpha_1, \dots, \alpha_m$ are linearly independent. Hence, $\beta_1 = \dots = \beta_m = 0$. \square

5. MORE EXAMPLES

The reader can easily check that Theorem 6 can be applied to Examples 4 and 5. The next example concerns the estimation of the dimension of the eigenspace in Theorem 6. We will show that this dimension can be any number between 0 and $M/2$, where M is defined as in the proof of Theorem 2.

Example 13. Let $S = \mathbb{Z}_2^m$ with identical involution and let $u = (1, 0, 0, \dots, 0)$. We have $M = 2^m$. The dual semigroup S^* can be identified with $\{-1, 1\}^m$. There are 2^{m-1} characters σ on S satisfying $\sigma(u) = -1$ and 2^{m-1} characters σ satisfying $\sigma(u) = 1$. We denote those characters by $\sigma_1, \dots, \sigma_{2^{m-1}}$ and $\rho_1, \dots, \rho_{2^{m-1}}$, respectively. For fixed $k, l \in \{0, \dots, 2^{m-1}\}$ we put⁴ $\mu := \sum_{i=1}^k \delta_{\sigma_i} + \sum_{j=1}^l \delta_{\rho_j}$ and $\phi := \mathcal{L}(\mu)$. Since the support of the measure consists of $k + l$ points, the space \mathcal{H}^ϕ is $k + l$ dimensional. To see this one can use the interpretation of \mathcal{H}^ϕ as \mathcal{P}^μ , as in Example 4. Now let us observe that

$$(15) \quad \sigma_1, \dots, \sigma_k \in \ker(u_\phi + I), \quad \rho_1, \dots, \rho_l \in \ker(u_\phi - I),$$

by the same argument as in the proof of Theorem 6 (iii) \Rightarrow (ii). Since the characters are always linearly independent we get that $\dim \ker(u_\phi + I) \geq k$ and $\dim \ker(u_\phi - I) \geq l$. But the eigenspaces corresponding to -1 and 1 are orthogonal, thus $\dim \ker(u_\phi + I) = k$ and $\dim \ker(u_\phi - I) = l$.

⁴ We use the convention $\sum_{i=1}^0 a_i := 0$

Let us put $e = (0, 1, 0, \dots, 0)$ and take two numbers $l_1 \in \{0, \dots, 2^{m-1}\}$ and $l_2 \in \{0, \dots, 2^{m-2}\}$. Using the same technique we can construct a mapping $\phi \in \mathfrak{P}(S)$ such that $\dim \ker(u_\phi + I) = l_1$ and $\dim \ker(e_\phi + I) = l_2$.

In the following example there are three elements satisfying $2u = 0$ and $u = u^*$, with three different upper bounds for the dimensions of the kernel.

Example 14. Let us consider the semigroup $S = U(\mathbb{Z}_2^2, \mathbb{Z}_2, \pi)$ where $\pi(x, y) = x$ for $x, y \in \mathbb{Z}_2$. The involution on S is identity. Note that $(1, 0) + s \neq s$ for $s \in S$, but $(0, 1) + s \neq s$ only for $s \in \mathbb{Z}_2^2$. Hence, the upper bounds for the dimensions of the kernels are three and two, respectively. The dimension of the kernel for $(0, 0)$ is obviously zero.

Remark 15. Let us take two $*$ -separative semigroups S and T , both having neutral elements (0_S and 0_T respectively) and a $*$ -homomorphism $h : S \rightarrow T$ satisfying $h(0_S) = 0_T$. Take an element $u \in T$ satisfying $2u = 0_T$ and $u = u^*$. The $*$ -semigroup $U(S, T, h)$ has a zero, namely 0_S . However, the element u , understood as an element of $U(S, T, h)$, does not satisfy the condition $2u = 0_{U(S, T, h)}$. Nevertheless, we have $3u = u$ and $u^* = u$, which by [15] guaranties boundedness (and hence selfadjointness) of u_ϕ for any $\phi \in \mathfrak{P}(U(S, T, h))$. The indefinite inner product space $(\mathcal{H}^\phi, \langle u_\phi \cdot, - \rangle)$ is then degenerate, i.e. u_ϕ has a non-trivial kernel.

We could investigate, instead of positive definite mappings on S , the set of positive definite forms on S . Namely, we say that $\phi : S \times \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{C}$ is a form over (S, \mathcal{E}) if for every $s \in S$ the mapping $\phi(s, \cdot, -)$ is a hermitian bilinear form on the linear space \mathcal{E} . We say that a form is positive definite if for every finite sequences $(s_k)_k \subseteq S$, $(f_k)_k \subseteq \mathcal{E}$ we have $\sum_{i,j} \phi(s_j^* + s_i; f_i, f_j) \geq 0$. For a positive definite form ϕ we can construct a Hilbert space \mathcal{H}^ϕ which together with the functions $K_{s,f}^\phi$ ($s \in S$, $f \in \mathcal{E}$) constitute a RKHS. Like in the case of $\mathcal{E} = \mathbb{C}$, cf. [15] and also [17], we can define the (closed) shift operator associated with an element $u \in S$ by $u_\phi(K_{s,f}^\phi) = K_{s+u,f}^\phi$. The following example shows, that in this setting the equivalence in Theorem 6 no longer holds.

Example 16. Let $S = \mathbb{Z}_2$ (with the identical involution) and let \mathcal{E} be an infinite dimensional Hilbert space. Consider the following form

$$\phi(x, f, g) = \begin{cases} \langle f, g \rangle_{\mathcal{E}} & : x = 0 \\ \langle -f, g \rangle_{\mathcal{E}} & : x = 1 \end{cases}.$$

Note that

$$\sum_{x,y=0,1} \phi(x+y, f_x, f_y) = \langle f_0, f_0 \rangle + \langle f_1, f_1 \rangle - 2\Re \langle f_1, f_0 \rangle = \|f_1 - f_0\|^2$$

which is greater or equal to zero for any choice of $f_0, f_1 \in \mathcal{H}$.

The space \mathcal{H}^ϕ can be realized as $\mathcal{H}^\phi = \mathcal{E}$ so as $K_{0,f}^\phi = f$ and $K_{1,g} = -g$, $f, g \in \mathcal{E}$.

Take $u = 1$. The condition (i) of Theorem 6 is satisfied because the semigroup is of finite cardinality. On the other hand $u_\phi K_{0,f}^\phi = K_{1,f}^\phi = -f$ for $f \in \mathcal{H}$. Hence, $\dim \ker(u_\phi + I) = \dim \mathcal{H} = \infty$.

6. FINAL REMARKS

Our work is connected in a way with many other papers and books. Let us mention some of them.

- The transformation $\phi \mapsto \phi(\cdot + u)$ has been investigated by Bisgaard. He showed in [9] that it is always a sum of four positive definite mappings.
- Functions with finite number of negative spaces on topological groups has been considered in the book of Sasvári [14].
- In [1] sequences on \mathbb{N} with a finite number of negative squares are considered.
- In [2] the authors consider negative definite sequences, which is a subclass of mappings with a finite number of negative squares.
- Finally, in [10] definitizing ideals are investigated.

REFERENCES

- [1] C. Berg, J.P.R. Christensen, P.H. Maserick, Sequences with finitely many negative squares, *J. Funct. Anal.* **79** (1988), 260–287.
- [2] Ch. Berg, J.P.R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups*, Springer-Verlag, New-York Berlin Heidelberg Tokyo **1984**.
- [3] J. Bognár, *Indefinite Inner Product Spaces*, Springer-Verlag **1974**.
- [4] T.M. Bisgaard, On the Relation between the Scalar Moment Problem and the Matrix Moment Problem on $*$ -semigroups, *Semigroup Forum* **68** (2005), 25–46.
- [5] T.M. Bisgaard, Two sided complex moment problem, *Ark. Mat.* **27** (1989), 23–28.
- [6] T.M. Bisgaard, Separation by characters or positive definite functions, *Semigroup Forum* **53** (1996), 317–320.
- [7] T.M. Bisgaard, Extensions of Hamburger’s Theorem, *Semigroup Forum* **57** (1998), 397–429.
- [8] T.M. Bisgaard, Semiperfect countable \mathbb{C} -separative C -finite semigroups, *Collect. Math.* **52** (2001), 55–73.
- [9] T.M. Bisgaard, Factoring of Positive Definite Functions on Semigroups, *Semigroup Forum* **64** (2002), 243–264.
- [10] T.M. Bisgaard, H. Cornean, Nonexistence in General of a Definitizing Ideal of the Desired Codimension, *Positivity* **7** (2003), 297302.
- [11] D. Cichoń, J. Stochel, F.H. Szafraniec, Extending positive definiteness, preprint
- [12] A.H. Clifford, G.B. Preston, *The Algebraic Theory of semigroups* 2 vols., AMS, Providence, **1961, 1967**.
- [13] I.S. Iohvidov, M.G. Krein, H. Langer, *Introduction to the Spectral Theory of operators in Spaces with an Indefinite Metric*, Akademie-Verlag, Berlin **1982**.
- [14] Z. Sasvári, *Positive Definite and definitizable functions*, Akademie Verlag, Berlin **1994**.
- [15] F.H. Szafraniec, Boundness of the shift operator related to positive definite forms: an application to moment problems, *Ark. Mat.* **19** (1981), 251–259.
- [16] F.H. Szafraniec, Interpolation and domination by positive definite kernels, in *Complex Analysis - Fifth Romanian-Finish Seminar*, Part 2, Proceedings, Bucarest (Romania), 1981, eds. C. Andrean Cazacu, N. Boboc, M. Jurchescu and I. Suciu, Lecture Notes in Math., **1014**, 291–295, Springer, Berlin-Heidelberg, 1983.
- [17] F.H. Szafraniec, The Sz.-Nagy ”théorème principal” extended. Application to subnormality, *Acta Sci. Math. (Szeged)*, **57**(1993), 249–262.
- [18] F.H. Szafraniec, The reproducing kernel Hilbert space and its multiplication operators, *Oper. Theory Adv. Appl.*, **114**(2000), 253–263.
- [19] B. Sz.-Nagy, *Extensions of linear transformations in Hilbert space which extend beyond this space*, Appendix to F. Riesz, B. Sz.-Nagy, *Functional Analysis*, Ungar, New York, **1960**.
- [20] M. Thill, Exponentially bounded indefinite functions, *Math. Ann.* **285** (1989), 297–307.

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